$$\cdot J_n\left(\frac{\lambda'_{np}}{b}\rho\right)J_n\left(\frac{\lambda'_{nq}}{b}\rho\right)\Bigg]\rho\,d\rho$$

$$= \begin{cases} 0 & p \neq q \\ \frac{b^2}{2}\left(1-\frac{n^2}{\lambda'^2_{np}}\right)J_n^2(\lambda'_{np}) & p = q \end{cases} \quad (42)$$

where λ_{np} and λ'_{np} are the zeros of ordinary Bessel functions and their derivatives, respectively. These integrals appear, respectively, in orthogonality relationships between TM and TE modes in homogeneously filled waveguides, i.e.,

$$\iint E^{TM}_{t_i} \cdot E^{TM}_{t_j}\, ds = 0 \qquad i \neq j \quad (43)$$

and

$$\iint E^{TE}_{t_i} \cdot E^{TE}_{t_i}\, ds = 0 \qquad i \neq j\,. \quad (44)$$

V. Summary

Indefinite Bessel function integrals useful in solving electromagnetics problems in lossy media with circular symmetric geometry have been presented for ordinary Bessel functions of the first and second kinds, for modified Bessel functions, and for combinations of ordinary and modified Bessel functions. Two orthogonal definite integrals have been presented for Bessel functions of the first kind. Additionally, six recurrence identities for similar combinations of Bessel functions have been presented. Limiting values of the integrals at zero and infinity have been given to facilitate their use in practical application. These integrals and recurrence identities are useful in any analysis that deals with products of entities which are solutions to the Helmholtz equation in cylindrical coordinates.

References

[1] D. Kajfez, "Indefinite integrals useful in the analysis of cylindrical dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 873–874, Sept. 1987.
[2] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions.* New York: Dover, pp. 479–494.
[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products.* San Diego: Academic Press, 1980, pp. 633–634.
[4] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions.* New York: Dover, p. 358.
[5] ——, *Handbook of Mathematical Functions.* New York: Dover, pp. 361, 376.

Theory for a Cylindrical Pillbox Accelerator Cavity Using Layered Structures for Reducing Skin-Effect Losses

W. C. Sailor, F. M. Mueller, and B. E. Carlsten

***Abstract*—It is shown that for a cylindrical pillbox accelerator cavity operating in a TM_{0n0} mode, the use of laminated conductors for the flat walls in conjunction with a multilayered dielectric structure for the round walls can decrease skin-effect losses by an order of magnitude over that of a copper cavity having the same accelerating field. The layered dielectric structure for the round walls works in a fashion similar to a quarter-wave interferometer. The laminated conductor on the flat walls reduces the ohmic losses by effectively increasing the skin depth.**

I. Introduction

Accelerator designs for nuclear and particle physics and free-electron laser applications have, in general, been either room-temperature copper or superconducting. Superconducting accelerators, besides having problems such as higher order mode dissipation, require the use of exotic fabrication techniques and the high cost and complexity of cryogenic systems. Acceleration gradients or duty factors in room temperature systems may be limited by the power loss, $P_{\rm loss}$. A fair measure of the importance of wall losses in different cavity designs is the shunt impedance

$$R_s = \frac{\left(\int E_z\, dz\right)^2}{P_{\rm loss}} \quad (45)$$

where E_z is the field seen by a particle undergoing acceleration in the cavity. Here we describe the theoretical calculations of the effects of two methods of material layering on $P_{\rm loss}$ in a cylindrical pillbox cavity, while keeping the same acceleration field. The outer (round) wall of the cavity interacts with the fields in a way similar to layered optical coatings [1]. Thus, to greatly reduce the ohmic losses in the round walls we use a set of concentric annuli of alternating high and low dielectric materials, backed by a metal substrate. For the flat walls, the situation is analogous to a shielded coaxial transmission line, for which the layered metal/dielectric structure of [2] has been found to be useful. Such a structure works by effectively increasing the penetration of the fields into the conducting material, and may be understood in terms of an increase in the classical skin depth [3].

II. Cavity Field Description

The electromagnetic fields are taken to satisfy the wave equation, with the assumption of a constant dielectric constant ε and magnetic permeability ($\mu = \mu_0$), and zero net charge density everywhere

$$\nabla^2\psi = \mu_0\varepsilon\,\frac{\partial^2\psi}{\partial t^2} + \mu_0\sigma\,\frac{\partial\psi}{\partial t}\,. \quad (46)$$

The variable ψ is any of the components of $H, E,$ or D.

We will here analyze a TM_{0n0} mode, where n refers to the number of radial nodes of the electric field. The electric field thus points along the axis of the cavity and the magnetic fields are strictly circular.

The geometry of the system is described in Fig. 1. There is layering in both the radial and the longitudinal directions. The geometry is such that the boundaries between layers are all described by cylinders concentric with the z axis or planes of constant z. The flat layers that comprise the cavity wall consist of alternating metal, dielectric,

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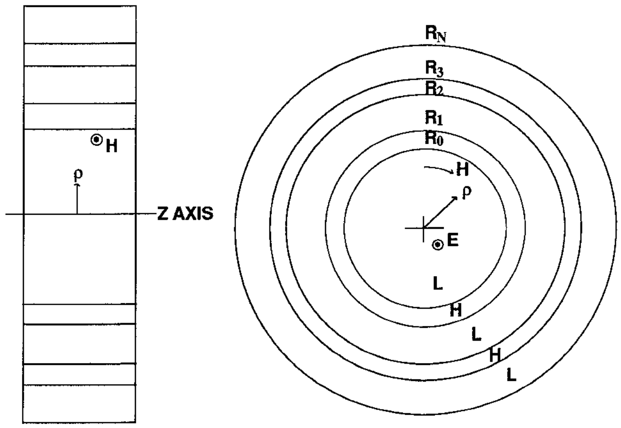

Fig. 1. Layering concept for the cavity. The beam axis is the z axis. The layers are dielectrics of alternating high (H) , low (L) dielectric constant. There is also very fine (typically 0.2 μm) layering in the flat walls.

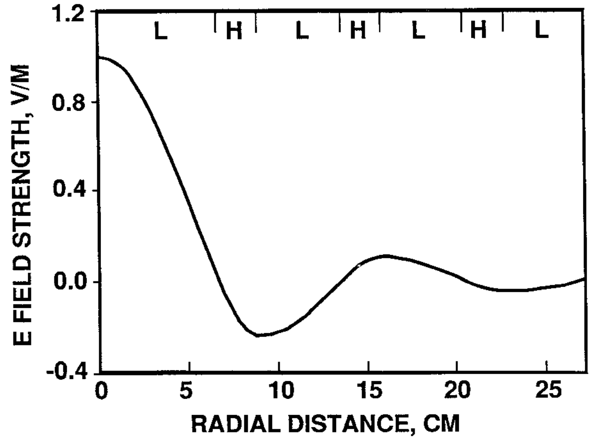

Fig. 2. The effect of the addition of three pairs of dielectric layers on the electric field strength. The tick marks at the top of each graph show the interface locations.

metal, etc. The central accelerating region is for our purposes here a dielectric region. The outer round walls are of alternating high, low, high dielectric material. Thus, the entire cavity consists of a connected set of annuli that are filled with either dielectric material or metal. The exception is that the innermost regions are cylinders.

With the usual separation of variables we write:

$$D_z = F(\rho)G(z)e^{j\omega t}, \tag{47}$$

$$F(\rho) = a_n J_0(\beta_n \rho) + b_n Y_0(\beta_n \rho), \tag{48}$$

$$G(z) = A_k e^{\zeta k z} + B_k e^{-\zeta k z} \tag{49}$$

for the nth radial section and the kth axial section. The variables β_n and ζ_k are separation constants and the a_n, b_n, A_k, and B_k are constants to be determined. The TM_{0n0} modes in a pillbox have no variation in z, and thus:

$$\zeta_0 = \left[\beta_0^2 - \mu_0 \varepsilon \omega^2 + \mu_0 \sigma i \omega\right]^{1/2} = 0. \tag{50}$$

Because there is no conductor in the central region, we have

$$\beta_0 = [\mu_0 \varepsilon]^{1/2} \omega = \kappa_0^{1/2} \omega / c \tag{51}$$

where κ_0 is the ratio of the dielectric constant in region zero to that of free space. In addition, because $E_z = E_{\text{cav}}$ = a constant in the central region, then $a_0 = 1$, $\kappa_0 A_0 = E_{\text{cav}}$, and $B_0 = 0$.

III. Multilayered Dielectrics for the Round Walls

The stack of dielectrics plus the metal backing operates like the multi-layered structure described in [2]. A microwave analogy is a quarter-wave impedance-matched radial transmission line terminated by round walls.

There are N layers of dielectric of alternating high and low dielectric constant. The layer regions have indexes $n = 1, 2, \cdots N$. The central accelerating region has the index $n = 0$ and also contains a dielectric. The outer radius of each of the N regions has the symbol R_n. Each region has its own separation constant β_n, which has the following value:

$$\beta_n = [\mu_0 \varepsilon]^{1/2} \omega = \kappa_n^{1/2} \omega / c. \tag{52}$$

The longitudinal index k is 0 and $\zeta_0 = 0$, from (6). The coefficients a_n and b_n are the only part of the solution (3) that varies from region to region. There are $N + 1$ regions, when the central region is included. It is required that the value of E_z and H_ϕ (or $\partial E_z / \partial \rho$) be continuous.

We set a zero of the electric field amplitude for a transition to a high-dielectric layer and a zero of the magnetic field amplitude (zero electric field derivative) for the transition of the low dielectric layer. This approach is consistent with the optical plane wave case, and allows the cavity design to occur in a step-by step fashion.

The F function in the first, inner-most region is fixed, with $b_0 = 0$ and $a_0 = 1$. A computer program then steps forward in ρ and finds either the next root of (4), if the transition is to be from a high to a low dielectric, or the next root of the derivative of (4), if the transition is of the vice-versa type. This determines the radius for the transition. At the transition, there are two equations (continuity of electric field and its derivative) in two unknowns (the new a and b). These equations are solved and the procedure repeated again for the next layer.

In this fashion, the electric and magnetic fields can be found for any number of layers. The losses into the metal at the outmost portion of the last layer can be calculated from classical skin-depth theory, where the power dissipation per unit area is proportional to the square of the magnetic field strength. Figs. 2 and 3 show the results of such calculations for three pairs of dielectric layers. The frequency is 1.0 GHz. The high and low dielectric constants κ_H and κ_L equal 10 and 3, respectively. The innermost region is filled with dielectric of value $\kappa_L = 3$. It is seen that the effect of the layers is to cancel the electric field in such a way that its gradient near the metal wall is greatly decreased.

There is a node in the electric field wherever the dielectric changes from the higher to the lower value. The number of nodes is $N/2 + 1$, and the mode is therefore $TM_{0,N/2+1,0}$.

The code was used to calculate the losses in the round walls of a cavity versus number of pairs of dielectric layers. For the chosen frequency and dielectric constant values described previously, a semilogarithmic relationship was found for the round wall losses with N. The relationship between relative losses on the round walls and N (for even N) is approximately

$$\frac{P}{P_0} = 0.554^N. \tag{53}$$

Since the magnetic field strength is greatly reduced at the outer round wall, the losses there are reduced, even though the total area for the losses increased with number of pairs. The number 0.554 is very close to the ratio of the indices of refraction of the two dielectrics used: 0.548.

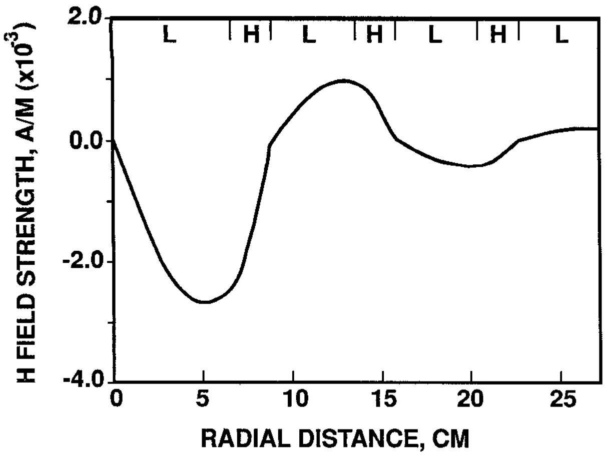

Fig. 3. Similar to Fig. 2, except showing the magnetic field strength. The value at the outer wall is greatly reduced compared to the first interface location.

We have performed calculations using the finite-difference code URMEL-T [4] that have confirmed the effectiveness of the layered dielectric structure and the accuracy of our analytical formulas.

IV. Layered Structure for the Flat Walls

Without doing anything about the losses in the flat walls, there would only be a slight decrease in the overall cavity wall losses. The layers that comprise the flat cavity walls consist of alternating metal, dielectric, metal, etc. The value of the dimensionless dielectric constant used in the layers will be different than that in the central region, and will have the value κ_d. The central region ($k = 0$) is a dielectric region. For the discussion here, we will look at a fixed value of the index n, i.e., we will be restricted to a range of ρ value where the value of κ in the central region is constant, and equal to κ_n.

In order to illustrate how the multilayered structure works, the general solution (2) is manipulated into a form similar to the equations of [3]. From (2) and Faraday's law the value of H_ϕ is:

$$H_\phi = -\frac{\beta_n(a_n J_1(\beta_n\rho) + b_n Y_1(\beta_n\rho))\left(A_k e^{\zeta_k z} + B_k e^{-\zeta_k z}\right)e^{j\omega t}}{i\omega\varepsilon\mu_0} \tag{54}$$

for the region $R_{n-1} < \rho < R_n$, and satisfies

$$\frac{\partial^2 H_\phi}{\partial z^2} = \zeta_{k^2} H_\phi \tag{55}$$

where,

$$\zeta_k = \left[\beta_n^2 - \mu_0\varepsilon\omega^2 + \mu_0\sigma i\omega\right]^{1/2}. \tag{56}$$

From Ampere's Law

$$-\frac{\partial H_\phi}{\partial z} = \frac{\partial D_\rho}{\partial t} + D_\rho\frac{\sigma}{\varepsilon} = \left(i\omega + \frac{\sigma}{\varepsilon}\right)D_\rho = (i\omega\varepsilon + \sigma)E_\rho. \tag{57}$$

Since E_ρ is parallel to the interfaces between boundaries the electric field will be used instead of the displacement in this discussion. The last three relationships are combined. Only the term in (12) containing the conductance is important in the metal regions. In the dielectric layers, only the other two terms are important. If we let the thickness of a very thin metal layer be W and the thickness of a very thin adjacent dielectric layer be T, and the magnetic field strength at the interface be $H_{\phi,I}$, the change in E_ρ is given to first order as

$$(\Delta E_\rho)_{\text{metal}} \approx i\omega\mu_0 H_{\phi,I} W \tag{58}$$

$$(\Delta E_\rho)_{\text{dielectric}} \approx i\omega\mu_0\left(1 - \frac{\kappa_n}{\kappa_d}\right)H_{\phi,I}T. \tag{59}$$

Therefore, when $\kappa_n = \kappa_d(1 + W/T)$, the change across the metal is just balanced by the change across the dielectric lamina in this approximation. This is the basic reason for the deep penetration of the fields into the laminated structure. The layering procedure will only be beneficial if $\kappa_n > \kappa_d$, i.e., the central region cannot be simply a vacuum region. For $\kappa_n = \kappa_d$, the attenuation will be unchanged from that of a purely metal wall.

In order to actually calculate the fields within the walls, a procedure similar to that of Clogston [3] is used. The problem is to find the coefficients A_k and B_k for $k = 1, 2, 3, \cdots, K$, where K is the total number of layers (an even number). Beyond the last layer there is a metal substrate region where $A_{K+1} = 0$. There are thus $2K + 1$ unknowns. There are K interfaces, with two equations per interface; one for the continuity of H_ϕ, and the other for the continuity of E_ρ. The last equation is that at the left hand edge of region 1, H_ϕ must reduce to the fixed value

$$H_{\phi,0} = -\frac{\beta_n(a_n J_1(\beta_n\rho) + b_n Y_1(\beta_n\rho))E_{\text{cav}}e^{j\omega t}}{i\omega\varepsilon\mu_0}. \tag{60}$$

A FORTRAN program was written to evaluate the wall losses as a function of W, T, K, and the κ's. The continuity of E and H lead to a matrix of m equations in m unknowns, where $m = 2K + 1$.

For a 1 GHz cylindrical pillbox cavity simulation we chose the cavity dielectric constant κ_d equal to 3, the layer dielectric constant $\kappa_n = 2.25$ and the surface H to be equal to 1.00 A/m. Under these conditions, the classical skin depth is 2.1 μm. The ratio of layer thicknesses, W/T, is set at 3, so that the condition $\kappa_n = \kappa_d(1 + W/T)$, needed to optimize the enhancement, is met. The resistivity is that of pure copper.

A dramatic improvement in shunt impedance [(1)] versus a metal cavity can be said to be obtained when a factor of 10 drop in wall losses has been attained. In order to establish how many pairs of layers will be needed for this, (94) of Clogston was used. Stating this equation in terms of our variables

$$\frac{P}{P_0} = \frac{W}{\delta\sqrt{3}} \tag{61}$$

where P is the power loss to the wall per unit area, P_0 is the value of P for pure metal walls, and δ is the skin depth. It is implied that for a factor of 10 improvement, the metal layer thickness W should be 17.3% of the skin depth. The new "skin depth" will be ten times that of plain metal. Therefore, for each new skin depth, $10/0.173 = 58$ metal layers will be needed. To achieve complete effectiveness, probably at least three skin depths worth of layering will be needed. Thus, 200 pairs of layers were used.

The computer code was run with the above inputs. The resulting real part of the magnetic field is shown in Fig. 4. The penetration of the H field into the structure is a smooth function that resembles an exponential. The exponential drops to $1/e$ of its value in about 90 μm of layering. The effective skin depth for this structure is 22.5 μm of metal, because only a quarter of the structure is metal. Thus, a factor of roughly ten improvement has been achieved.

V. Conclusions

Combining the layered dielectric ($\kappa = 10, 3$; three pairs) structure for the round walls with 200 layers of metal/dielectric ($\kappa = 2.25$) pairs on the flat walls results in about a factor of 18 reduction in skin effect losses and improvement in R_s. These results, if they were achieved in a useful accelerator cavity, would be quite spectacular.

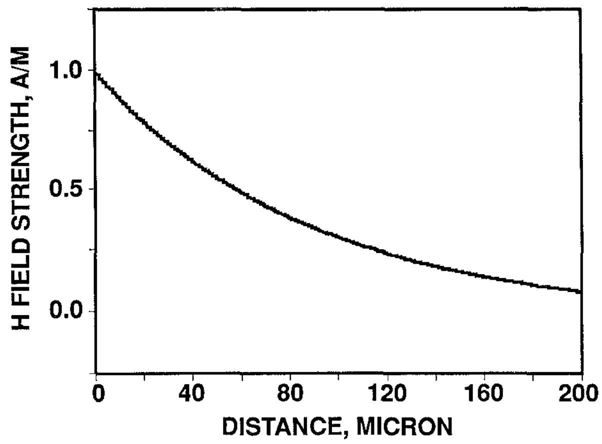

Fig. 4. The variation in the real part of the magnetic field strength through the flat-wall layering. There is 17.3% of a skin depth per layer, and 200 pairs of layers. This is to be compared to a skin depth in plain metal of 2.3 μm.

The problem of dielectric breakdown on the surfaces of the annular dielectric regions may be great practical trouble. This is exacerbated by the fact that in accelerating structures these surfaces will tend to absorb residue gas and stray ions from the surrounding environment.

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References

[1] M. Born and E. Wolf, *Principles of Optics, Electromagnetic Theory of Propagation, Interference, and Diffraction of Light.* 6th ed., Oxford: Pergamon, 1980.
[2] A. M. Clogston, "Reduction of skin-effect losses by the use of laminated conductors," *Proc. I. R. E.*, vol. 39, no. 7, July 1951.
[3] J. D. Jackson, *Classical Electrodynamics.* 2nd ed., New York: Wiley, 1975.
[4] U. Laustrier, U. Van Rienen, and T. Weiland, "URMEL and URMEL-T user guide (Model analysis of cylindrically symmetric cavities; Evaluation of RF-fields in waveguides," DESY M-87-03, Feb. 1987.